Chapter 3

Uniform Regimes in Financial Practice

3.1. Preliminary comments

In this chapter we will consider financial laws widely applied in the practice of investment and discount. One of their common features is the *uniformity in time,* so that the calculation of accumulated and discounted values depends only on the duration of the operation.

It is clear that the return of an operation is measured by a per period rate[1]. In a uniform law, if the rate remains constant for all given periods (we then talk about *flat structure,* in the field of all possible term structures of interest rate, concepts that we will consider later), it is clear that percentage returns remain unchanged wherever the operation is located in the time axis. This does not happen in financial markets, where to be at least approximately realistic, it would be necessary, in order to keep the simplicity of uniform law, to use the flat structure for a relatively short period. If this cannot be done because of the variability of returns with time, it is necessary to use the laws of two variables, characterized by per period rates changing with current time.

We will consider three couples of *uniform financial regimes*[2] that give rise to many infinite families of uniform laws of interest and discount identified by the return parameters.

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1 Let us recall that the per period rate measures the price for the availability of money in the given period.

2 When we distinguish between accumulation and discount, instead of "regime" we can talk about "couple of regimes" of interest and discount.

Because of uniformity we can use 0 as the initial time of the operation of investment (or the maturity of discounts), using small letters for duration (see section 2.5).

3.1.1. *Equivalent rates and intensities*

In any given financial regime of interest or discount, the problem of comparing rates or interest relative to different duration often arises. The following definitions hold.

Two per period interest (or discount) rates for different durations are said to be *equivalent* if they give rise to the same percentage of annual return and then, according to previous definitions, if they follow from the same financial law of interest (or discount).

Two intensities of interest (or discount) for different durations are said to be *equivalent* if they correspond to equivalent rates, and then if they follow from the same financial law of interest (or discount).

Two per period rates, one of interest for the length t' and the other of discount for the length t'', are said to be *equivalent* if they give rise to returns expressed by the annual interest and discount corresponding to conjugate laws. The equivalence for intensities follows from the equivalence for per period rates.

Rates and intensities for the regimes, discussed in the following text, are to be considered “initial”.

3.2. The regime of simple delayed interest (SDI)

Continuing the considerations in section 1.1, we observe that the simplest way to calculate interest on a loan amount C is to consider the interest I proportional both to the principal C and the duration $t = y-x$ (with no dependence on the initial time x) obtained as:

$$I = C\, i\, t \tag{3.1}$$

Parameter i, which is usually given in percentage form $r\% = r/100$, where $r = 100\, i$, measures the interest for a unitary capital and a unitary time interval. Assuming from now on (unless otherwise stated) that the year is the unit measure for time, i is called the *annual interest rate (delayed)*. The accumulated amount $M = C+I$ after time t is then given by

$$M = C\,(1 + i\,t) \tag{3.2}$$

Relations (3.1) and (3.2) for each choice of C, i, t, are characteristic of the *regime of simple delayed interest (SDI)*, in which interests are paid, or booked, only at the end of the loan of length t.

It follows from (3.1) and (3.2) that for accumulation laws in the SDI regime

– the *accumulation factor* for the length $t = y - x > 0$ is

$$u_t = 1 + i\,t \tag{3.3}$$

– the *per period interest rate* (1.3) for the length t is

$$i_t = i\,t \tag{3.4}$$

– the *per period interest intensity* for the length t is

$$j_t = i_t / t = i \tag{3.5}$$

independent of the duration and equal to the annual interest rate[3].

Relation (3.4) gives the equivalent per period rates to a given annual rate i. More generally, for durations that are not alike and different from a year, there exists proportionality between equivalent per period rates and lengths. In symbols, if $i_{t'}$ and $i_{t''}$ are the rates for the length t' and t'', they are equivalent if

$$i_{t'} / t' = i_{t''} / t'' = I \;^{4} \tag{3.6}$$

EXAMPLE 3.1.– If in the SDI regime the quarterly interest ($t' = 1/3$) is 5.25%, the equivalent semi-annual rate ($t''= 1/2$) is: 0.05253/2 = 0.07875, or 7.875%. They both give rise to the annual return $i = 0.1575$ which also measures the intensity.

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3 Such equality, which is only numeric and not dimensional, is due to the fact that the interest rate is measured annually.

4 $\forall (t',t'')$ members in (3.6) are equal to the per period intensity and then the equivalence of rates gives the equivalence of the per period intensities between them and to i. "per period" refers to the period in which the return matures.

If the intensity changes during the lifetime of the loan, assuming the values $i^{(1)}, \ldots, i^{(n)}$ for the length $t_1,...,t_n$, (where $\sum_{s=1}^{n} t_s = t$), (3.2) can be generalized as:

$$M = C\,(1 + \sum_{s=1}^{n} i^{(s)} t_s) = C\,(1 + \bar{i}\,t) \tag{3.7}$$

where $\bar{i}$ is the arithmetic mean of intensities $i^{(s)}$ weighted with the length t_s.

EXAMPLE 3.2.– We invest €150,000 in the SDI regime obtaining for the first 3 months the annual interest (= intensity) of 5%, for the next 4 months interest of 5.5%, and for the next 6 months interest of 5.2%. The accumulated amount at the end is:

$$M = 150{,}000\cdot[1 + (0.05\cdot 3+0.055\cdot 4+0.052\cdot 6)/12] = €158{,}525$$

Exercises on the SDI regime

3.1

Calculate in the SDI regime the interest earned for 6 months on a principal of €1,500,000 at the annual rate of 8.25%.

A. Applying (3.1): I = €61,875

3.2

Calculate in the SDI regime (adopting bank year, with 360 days and each month having 30 days) the accumulated amount of a loan of €2,500,000 and of length 2 years, 6 months and 25 days at the annual rate of 9.5%.

A. Applying (3.2): M = €3,110,243

3.3

Calculate the accumulated amount as in Exercise 3.2, applying the varying interests: 9.5% in the 1st year, 10.5% in the 2nd year, 9% in the 3rd year.

A. Applying (3.3):

$$M = 2{,}500{,}000\ (1 + 0.095 + 0.105 + 0.09\ 205/360) = €3{,}128{,}125$$

The average annual interest for the operation is 0.25125·360/925 = 9.778%.

3.3. The regime of rational discount (RD)

From the SDI laws we can deduce the conjugated discount laws that give rise to reciprocal factors. They fall within the *rational discount (RD)* regime. The discounted value C, payable in x instead of the amount M at maturity $y>x$, is obtained from (3.2), resulting in

$$C = \frac{M}{1 + i\,t} \tag{3.8}$$

Giving the annual interest rate i of the conjugate SDI law, we obtain the RD law for which:

– the *discount factor* for the length t is

$$v_t = 1/(1 + i\,t) \tag{3.9}$$

– the *per period discount rate* for the length t is

$$d_t = \frac{i\,t}{1+i\,t} \tag{3.10}$$

– the *per period discount intensity* for the length t is

$$\rho_t = d_t\,/t = \frac{i}{1+i\,t} \tag{3.11}$$

If the annual discount rate $d = i/(1+i)$ is given[5], from which $i = d/(1-d)$, the previous quantities (3.9), (3.10) and (3.11) are obtained as a function of d:

$$v_t = \frac{1}{1-\dfrac{d\,t}{1-d}} = \frac{1-d}{1-d(1-t)} \tag{3.9'}$$

$$d_t = 1 - v_t = \frac{d\,t}{1-d(1-t)} \tag{3.10'}$$

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5 This law has a trivial interpretation: i is the interest paid at the end of the year on the unitary capital, while d is the discount or the interest paid at the beginning of the year. Then d is the discounted value of i, the relation is obtained from (3.10) posing $t = 1$. It is useful to make use of such arguments based on the financial equivalence's principle.

$$\rho_t = d_t / t = \frac{d}{1-d(1-t)} \tag{3.11'}$$

Equation (3.10') gives the per period discount rate for the length t equivalent to the annual discount rate d.

EXAMPLE 3.3

1) If in the RD regime the delayed interest i = 7.40%, using (3.10) we obtain the semi-annual, four-monthly, quarterly and monthly discount rates: 3.5680%, 2.4073%, 1.8164% and 0.6129%.

2) If in the RD regime the advance rate is d = 6.80%, using (3.10') we obtain the semi-annual, four-monthly, quarterly and monthly discount rates: 3.5197%, 2.3743%, 1.7914% and 0.6043%.

3) If in the RD regime the four-monthly discount rate is $d_{1/3}$ = 2.15%, inverting (3.10) with t = 1/3 we obtain the equivalent annual rate $i = 3 \cdot 0.0215/0.9785 = 0.065917$. Then the equivalent semi-annual rate $i_{1/2}$ is obtained through (3.10) and it is 3.1907%.

The amount D of the discount on M and the discounted amount C as a function of d are given respectively by

$$D = M\, d_t = \frac{M\, d\, t}{1-d(1-t)}\ ;\ C = M\, v_t = \frac{M\,(1-d)}{1-d(1-t)} \tag{3.12}$$

In Figure 3.1, for an SDI law, the graph of $I = I(t)$ and $M = M(t)$ are shown (see (3.1) and (3.2)) as a function of t. Figure 3.2 shows, for an RD law, the graph $C = C(t)$.

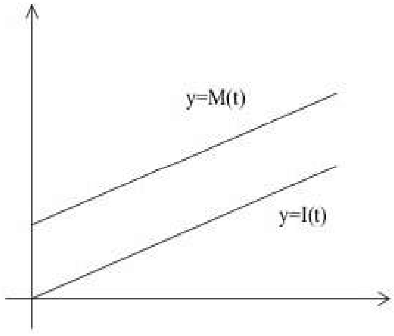

Figure 3.1. *Simple delayed interest*

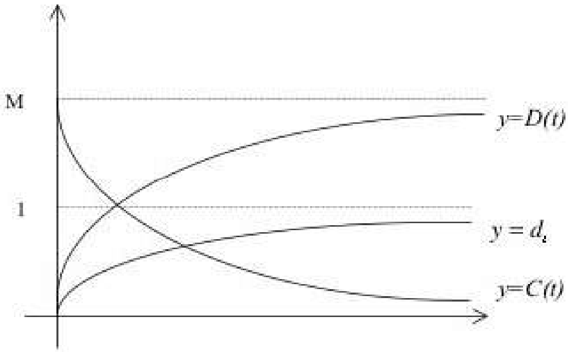

Figure 3.2. *Rational discount*

Comments

All linear laws, including conjugated laws, are used in general for short time periods. For the SDI laws, by indicating as g the number of days in the financial operation, the interest can be written as

$$I = \frac{C\,g}{T/i} \tag{3.13}$$

where T=360 if the "bank year" is used and T=365 if the "calendar year" is used. The numerator in (3.13) takes the name of "number" and the denominator that of "fixed dividend" because it depends only on the rate.

(3.13) is useful for finding the interest on a current account ruled by the SDI law in a given period (bank accounts are typical), because in order to calculate the interest in the considered period it is enough to sum the numbers relative to the days between two changes and divide by the fixed dividend.

Exercises on the RD regime

3.4

Calculate in the RD regime the discount to cash with a 3 month advance a credit of €30,000 at an annual interest rate of 6%.

A. By applying (3.10) and (3.12) with i = 0.06, t = 0.25, the following is obtained

$$d_t = 0.015/1.015 = 0.014778 = 1.4778\%$$

$$D = M\,d_t = €443.35$$

3.5

Calculate in the SD regime the discounted amount at 31 March of an amount of €160,000 payable on 31 August, following the calendar year and applying an annual discount rate of 6%.

A. By applying (3.12) with $d = 0.06$, $t = 153/365 = 0.419178$, $M = 160{,}000$, the following is obtained:

$$C = (160{,}000 \cdot 0.94)/[1-0.06(1-0.419178)] = €155{,}830.6.$$

3.4. The regime of simple discount (SD)

If in the choice of financial regime we consider the problem of discount and – with a symmetric argument that gave rise to the SDI laws – we want to find a regime that gives rise to proportionality between payment and terminal value and anticipation time, we obtain the *simple discount (SD)* regime. In the SD regime, the amount D of discount on a terminal value M for a length t is given by

$$D = M\, d\, t \tag{3.14}$$

Parameter d, which is usually given in percentage r%, where $r = 100\, d$, has the meaning of discount for unitary capital and for a unitary time interval and is called the *annual rate of discount*. The discounted amount $C = M\text{-}D$ at time x, corresponding to the amount M payable at maturity $y = x+t > x$, is given by

$$C = M\,(1 - d\,t) \tag{3.15}$$

From (3.14) and (3.15) it follows that, for a law in the SD regime,

– the *discount factor* for length t of advance is

$$v_t = 1 - d\,t \tag{3.16}$$

– the *per period discount rate* (1.4) for length t is

$$d_t = d \cdot t \tag{3.17}$$

– the *per period discount intensity* for length t is

$$\rho_t = d_t / t = d \tag{3.18}$$

independent of the length, and numerically equal to the given annual discount rate *d*.

As in the SDI regime with (3.4) and (3.5), (3.17) gives the per period discount rate equivalent to the annual rate *d*. More generally, there exists proportionality between equivalent per period rate and length, and

$$d_{t'} / t' = d_{t''} / t'' = d \tag{3.19}$$

results, so we have the independence of the discount intensity from length.

EXAMPLE 3.4.– If in the SD regime the bimonthly rate (t' = 1/6) is 1.25%, the equivalent semi-annual rate (t''= 1/2) is: 0.0125·6/2 = 0.0375 = 3.75%. Both give the percentage of advance annual return d = 7.50%.

Exercises on the SD regime

3.6

Let us assume that a bill of €3,500 has a deadline on 30 September of the year *T*. We ask for the discount at bank Z, in the SD regime at the annual rate of 7% with payment on 25 June of the same year. Not considering transaction costs, calculate the return.

A. Because of (3.15) it is given by

$$C = 3{,}500\,(1 - 0.07\,\frac{97}{360}) = €3{,}433.99.$$

3.7

It has been agreed on the anticipation at 20 May of the amount of €68,000 with maturity at 30 September of the same year, in the SD regime (using the calendar year) and fixing the four-monthly equivalent rate of 2.65%. Calculate the amount of discount.

A. The annual equivalent rate d is 0.0265·3 = 0.0795. Using (3.14):

$$D = 68{,}000 \cdot 0.0795\,\frac{133}{365} = €1{,}969.86$$

3.5. The regime of simple advance interest (SAI)

The interest law conjugated to the simple discount gives rise to the regime of *simple advance interest (SAI)*, which is also called the regime of *commercial interest.*

Using the annual advance interest rate d in an SAI law:

– the *accumulation factor* for length t is

$$u_t = 1 / (1 - d\,t) \tag{3.20}$$

i.e. inverse of the factor v_t defined in (3.16);

– the *per period interest rate (delayed)* for length t is[6]

$$i_t = \frac{d\,t}{1 - d\,t} \tag{3.21}$$

– the *interest intensity* for length t is

$$j_t = i_t / t = \frac{d}{1 - d\,t} \tag{3.22}$$

Multiplying (3.20) for a capital C invested in x the accumulated amount is obtained

$$M = C\,u_t \tag{3.20'}$$

at time $y = x+t > x$.

6 Equation (3.21) gives the per period interest rate equivalent to the advance annual rate d or to the delayed annual rate $i = d/(1-d)$ of the conjugate law.

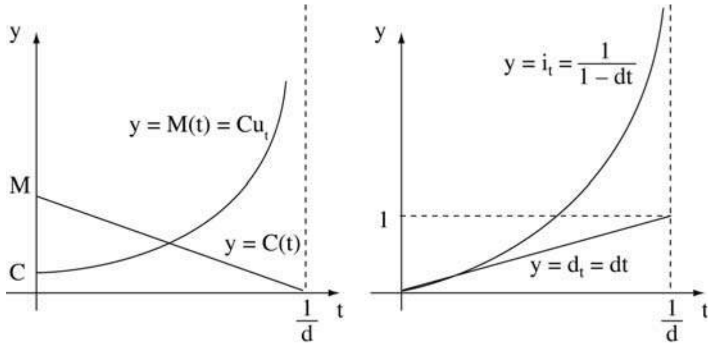

Figure 3.3. *a) Simple discount; b) simple advance interest*

EXAMPLE 3.5

1) In the SAI regime, given the advance rate d = 8.20%, the semi-annual, four-monthly, quarterly, monthly, etc., interest rate can be found using (3.21); 4.2753%, 2.8101%, 2.0929% and 0.6880% respectively are obtained.

2) In the SAI regime, given the delayed rate i = 9.50%, the corresponding rate d is 0.095/1.095 = 0.086758 = 8.6758%, and applying (3.21) the semi-annual, four-monthly, quarterly, monthly, etc., interest rate can be found; 4.5346%, 2.9781%, 2.2170% and 0.7282% respectively are obtained.

3) In the SAI regime, given the four-monthly interest rate $i_{1/3}$ = 2.35%, inverting (3.21) with t = 1/3 the equivalent annual rate d = 3·0.0235/1.0235 = 0.068881 can be found. Then the equivalent semi-annual rate $i_{1/2}$ can be found, using (3.21) to be 3.5669%.

Exercises on the SAI regime

3.8

Calculate the accumulated amount after 20 months of the investment of €120,000 in the SAI regime at the advance annual interest rate of 4.50%, and also the per period equivalent interest rate.

A. By applying (3.20) and (3.20') the following is obtained

$$M = 120{,}000/(1 - 0.045 \cdot 20/12) = €129{,}730.$$

The per period equivalent interest rate is calculated by (3.21) and the following is obtained:

$$0.075/(1 - 0.075) = 8.1081\%.$$

3.9

It is known that an 8 month discount operation in the SD regime at the annual rate d = 6% gives a discounted amount C = €155,000. Calculate:

– the capital at maturity;

– the per period discount rate;

– the per period interest rate in the conjugate law.

A. Given that the conjugate law to the applied SD law is an SAI:

– the capital at maturity is calculating using (3.20'):

$$M = 155{,}000/(1 - 0.06 \cdot 8/12) = €161{,}458;$$

– the per period discount rate is 0.06·8/12 = 4%;

– the per period interest rate in the conjugate law is 0.04/1.04 = 4.1667%.

3.10

Consider the same problem as in Exercise 3.9 but with: C = €155,000, d = 6% and t = 10.75 (= 10y+9m).

A. The capital at maturity is M = €436,620, the per period discount rate is 64.50% and the interest rate of the SAI law is 181.69%: note that the spread between the two rates increases. Note that the critical length threshold $t = 1/d$, such that the delayed interest and the accumulated amount diverge, is in this case 1/0.06 years=16 years and 8 months.

3.6. Comments on the SDI, RD, SD and SAI uniform regimes

Each of the two couples of uniform financial regimes considered in sections 3.2 and 3.3 and in sections 3.4 and 3.5 is made of a regime with factors which are linear functions of the length and another regime, which includes the conjugate laws, with factors which are a rational function of the length (their graph is an equilateral hyperbola). We can summarize this by saying that such regimes are made of uniform linear laws and their conjugate.

Let us summarize further properties of and observations about such couples.

3.6.1. *Exchange factors (EF)*

Using the symbols in section 2.4 we indicate by $g(\tau)$ the exchange factor (EF) for the length with sign τ (accumulation if $\tau>0$, discount if $\tau<0$) and we put $t=|\tau|$. If the corresponding laws are conjugate, (2.41) holds; then, $\forall\tau$, $g(\tau)$ and $g(-\tau)$ are reciprocal.

If we consider a couple of SDI and RD conjugate laws, we have, with $\tau = t > 0$:

$$g(\tau) = 1+i\,\tau = 1+i\,t \text{ (SDI)} ;\; g(-\tau) = 1/g(\tau) = 1/(1+i\,t) \text{ (RD)}$$

If we consider a couple of SD and SAI conjugate laws, we have, with $\tau = -t < 0$:

$$g(\tau) = 1+d\,\tau = 1-d\,t \text{ (SD)} ;\; g(-\tau) = 1/g(\tau) = 1/(1- d\,t) \text{ (SAI)}$$

3.6.2. *Corrective operations*

We notice, in the example of uniform financial laws considered here, that the operative role is similar to an "offsetting entry" that the conjugate laws have. Indeed, if an investment of C has been agreed with an SDI (or SAI) law for the length t, which gives rise to M, and to cancel such an investment, instead of an offsetting entry, we can restore the previous situation by applying to M the corresponding RD (or SD) factor.

3.6.3. *Initial averaged intensities and instantaneous intensity*

As already mentioned in footnote 3, values (3.5), (3.11), (3.18) and (3.22) are initial averaged intensities in the interval $(0,t)$ for investment or anticipation. The instantaneous intensity[7] in t (time from investment or time to maturity) has another meaning: it is obtained as a limit case of the continuing intensity defined in section 2.3.

Recalling that in the interest laws the instantaneous intensity are obtained from the logarithmic derivatives with respect to t of the exchange factors, the following expression for the instantaneous intensity in t can be easily deduced:

a) SDI (rate i):	$\delta_t = i/(1 + it)$	decreasing with t
RD (rate i'):	$\theta_t = i'/(1 + i't)$	decreasing with t
b) SAI (rate d):	$\delta_t = d/(1 - dt)$	increasing with t

7 Summarizing the definition in Chapter 2, in the accumulation of an investment made in x=0 the instantaneous intensity in $y=t>0$ is the limit of the per period intensity between y and $y+\Delta y$, with $\Delta y>0$, while in the discount of capital with maturity in x=0 the instantaneous intensity at time $y=-t<0$ is the limit of the per period intensity between y and $y+\Delta y$, with $\Delta y<0$.

SD (rate d') $\theta_t = d'/(1 - d't)$ increasing with t[8]

If in a) $i = i'$ or in b) $d = d'$, the corresponding laws are conjugate to each other.

3.6.4. *Average length in the linear law and their conjugates*

By applying the considerations in sections 2.5.2 and 2.5.3, it is easily verified that:

– in the *SDI regime:* the factor $u_t = 1+it$ is linear and then the average length $\hat{t}_q$ is the arithmetic mean of the investment length t_h, weighted with the amounts C_h. It can be verified that the equality between the interests $\sum_{h=1}^{n} C_h i t_h$, obtained with investments on times t_h, $(h=1,\ldots,n)$, and the interests $it\sum_{h=1}^{n} C_h$ obtained with only one investment for time t, can be obtained if and only if $t = \sum_{h=1}^{n} C_h t_h / \sum_{h=1}^{n} C_h$;

– in the *SD regime:* the factor $v_t = 1-dt$ is linear and then the average length $\hat{t}_q$ is the arithmetic mean of the discount length t_h, weighted with the amounts M_h;

– in the *SAI regime:* $\hat{t}_q$ is an associative mean of the length t_h, such that $1-d\,\hat{t}_q$ is the harmonic mean of the factors $1-dt_h$, weighted with the amounts C_h;

– in the *RD regime:* $\hat{t}_q$ is an associative mean of the length t_h, such that $1+i\,\hat{t}_q$ is the harmonic mean of the factors $1+it_h$, weighted with the amounts M_h.

3.6.5. *Average rates in linear law and their conjugated laws*

Referring to the symbols introduced in section 2.5.4 and using the same arguments used for the average length, we can deduce that:

– in the *regime SDI:* the average rate is the arithmetic mean of the rates i_h with weights given by the used amounts C_h;

– in the *regime SD:* the average rate is the arithmetic mean of the rates d_h with weights given by the capital at maturity M_h;

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8 The formal coincidence, due to the analytic properties of the exchange factors, of formulae (3.11) and (3.22) of initial intensity in the RD and SAI regimes with the respective instantaneous intensity does not change the difference between initial intensity, which is a domain function, and instantaneous intensity, which is a point function.

– in their conjugate regime, the average rates are obtained as associative means given by harmonic mean of the exchange factors, i.e. $1+i_h t$ with weights C_h in the RD regime and $1-d_h t$ with weights M_h in the SAI regime.

3.7. The compound interest regime

3.7.1. *Conversion of interests*

Let us reconsider the interest formation with an SDI law, which reflects a spontaneous propensity of the market due to the double proportionality of the interest with respect to the amount of the invested capital and also the length of investment, as (3.1) shows. However, we observe that if the interest is added to the principal at the end of the operation, then there is an asynchrony between the position of the *lender*, which gives his supply continuously (making it possible that other persons use his capital, depriving himself of its profitable use), and the *borrower*, who delays his payment until maturity. Such asynchrony, prejudicial for the lender, is greater the longer the time of investment. Thus, an investor can accept this regime, with equal return rates, only in the short-term (usually not longer than one year)[9].

Briefly, with the SDI regime the earned interest remains unprofitable until the end of the operation. Concerning SDI we can imagine the presence of two accounts: on the first account we book the principal C, giving interest with flow $C\,i$ and then with amount $C\,i\,\Delta t$ for every time of length Δt. However, such interest is booked on the second unprofitable account. At the end of the operation of length t the sums on both accounts, given by C and $I = Cit$, are withdrawn and transferred to the creditor[10]. It is then preferable to consider financial regimes that realize synchrony between the parties making the earned interest profitable. The transferring of earned interest between the unprofitable interest/account and the profitable principal/account, without having to wait until the capital is no longer being used, is called *interest conversion*. When these amounts are available for the creditor, he will be able to cash and use them elsewhere (and then the profitable capital in the original operation will remain unchanged) or he can add them to the capital (giving

9 A rough solution to the damage connected with the asynchrony can be obtained easily with an increment of the interest rate. Furthermore, the fair increment would increase with the length.

10 The SDI process is analogous to those of the following hydraulic scheme. A first tank holds a constant volume C of water; since time 0, by means of an open input tap some water flows into a second tank with a closed output tap; a gear is applied so that the inflow is proportional to C on the basis of the factor i, so we obtain a flow Ci. At time t the output tap is opened and the contents Cit of the second tank are poured into the first tank. All the water $C(1+it)$ is soon withdrawn.

more interest) in the same operation[11]. In this second case, a movement of money is not needed and it is enough to credit the interest in the same profitable principal/account.

It is obvious that intermediate conversions increase the amount, i.e. the lender credit, in t, as is shown below (considering, for simplicity, only one conversion). Let a principal C be invested at time 0 at annual rate i for the length t, with the assumption that the interest is formed using an SDI law but let the interest be converted at time $t_1 = t - t_2 < t$ and keep it invested at the same rate until t. Adding to C the interest Cit_1 earned at time t_1, the amount with the added interest becomes $M(t_1) = C(1+it_1)$ and the amount at term time t reaches the level: $M(t) = C(1+it_1)(1+it_2) = C[(1+it) + i^2 t_1 t_2]$. It is thus proved that an intermediate conversion increases, at the same interest, the final amount: the simple interest for time t_2 is added to the interest Cit_1 earned in the time t_1.

The compound interest regime is characterized by the conversion of simple interests to profitable capital during the operation.

Such a regime can be applied in two ways:

1) *the conversion is made with per period terms*, or more generally *in the discrete scheme*; this is the method used in bank and commercial practice, with conversion at the end of the calendar year, calendar quarter, etc. We will then talk about *discretely compound interest (DCI)*;

2) *the conversion is made continuously* over time, only in this case there is a perfect synchrony between the parties in the contract. We will then talk about *continuously compound interest (CCI).*

3.7.2. *The regime of discretely compound interest (DCI)*

A general approach to the DCI laws leads to the following scheme: the use of principal C for the length $t = t_1+t_2+ \ldots +t_n$ (using a year as measure of time, unless otherwise stated), such that in the sub-period of length t_s the intensities $i^{(s)}$, (s = 1, …, n) are used, and at the term of each sub-period the conversion is made. We then obtain the amount in t, given by

$$M(t) = C \prod_{s=1}^{n} (1+i^{(s)} t_s) \tag{3.23}$$

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11 The decision will depend on convenience and alternative uses and we will talk about this when discussing investment choices (see Chapter 4).

The product gives the accumulation factor from 0 to t in the DCI law[12].

Let us now consider some particular cases of discrete conversion that are relevant for banking and business application.

Accumulation with annual conversion

Assume in (3.23): $t_s = 1$, $\forall s$, then $t=n \in \mathcal{N}$ ($\mathcal{N}$ = set of natural numbers); $i^{(s)}$ = constant = i. A particular case of this model is for the conversion of interests at the end of the solar year. When we have only one payment C, made at the beginning of first year, the amount at the end of n^{th} year is given by

$$M(n) = C\,(1 + i)^n \tag{3.24}$$

EXAMPLE 3.6.– If €1,263,500 is banked at the beginning of 1998 in a bank account ruled by compound interests, annual conversion, at the annual rate of 4.35%, the terminal value at the 6th year (soon after the 6th conversion) is €1,631,285.

Mixed accumulation with annual conversion

With the hypothesis that the conversion is done on 31 December of each year, the amount $M(t)$ for the use of a principal C for a length t, in between n+2 years (i.e. the final part f_1 of the first year, other n years and the initial part f_2 of the $(n + 2)^{th}$ year, then $t = f_1 + n + f_2 < n+2$) is given by

$$M(t) = C\,(1 + i\,f_1)\,(1 + i)^n\,(1 + i\,f_2) \tag{3.25}$$

where the simple interest law is applied for a fraction of a year.

To maintain a bank account in which banking and withdrawal are made, we can apply the *direct method* making the algebraic sum of the relative amounts calculated using (3.25) from the time of movement until the common last time t. However, the *scalar method* is more often used, in which the “numbers” are found between subsequent balances in each calendar year and the conversion of interest is made at the end of the year or when the bank account is closed.

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12 Recalling that, due to the conversions, parameters $i^{(s)}$ are intensities and not also annual return rates, investing in 0 the principal C, the amount obtained after the 1st conversion is: $M(t_1) = C(1+i^{(1)}t_1)$ and becomes profitable with intensity $i^{(2)}$; then at the 2nd conversion we obtain: $M(t_2) = C(1+i^{(1)}t_1)(1+i^{(2)}t_2)$. And then, at the n^{th} conversion i.e. at time t, we obtain the result specified in (3.23).

EXAMPLE 3.7.– On 4 September 1996, Mr. John banks €23,500 on a bank account ruled by 4.65% per year, with mixed accumulation and annual conversion. The amount on 20 October 1999 is

$$M = 23500\ (1+0.0465\cdot 118/360)\ (1.0465)^2\ (1+0.0465\cdot 292/360) = €27{,}114.$$

Accumulation with fractional conversion

Let $\forall s$: $t_s = 1/m$ in (3.23), where $m\text{-}1 \in \mathcal{N}$; $i^{(s)}$ = constant = $j(m)$. We then have the conversion m times per year, where m is called *frequency* of the conversion of interest in profitable capital, indicating j as a function of the conversion frequency. This is the *intensity* parameter, where $K\ j(m)\ \Delta t$ is the interest for the profitable capital K for the length $\Delta t<1/m$. Parameter $j(m)$ is sometimes called the *nominal annual rate, convertible m* times a year or, more briefly, the *annual m/convertible rate*. The fractional conversion is usually used with the frequencies m = 2, 3, 4, 6, 12.

EXAMPLE 3.8.– If $m = 4$ (= quarterly conversion) and $j(4)$ = 8%/year, the interest on the capital K is $0.08\cdot K\cdot\Delta t$ for a period $\Delta t \leq 1/4$ and for a quarter the interest is $0.08\cdot 0.25\ K = 0.02K$. Using C for the capital at the beginning of the year, the amount at the end of the year (after 4 conversions) is

$$C\ (1 + 0.02)^4 = C\cdot 1.08243216$$

where the effective annual return is measured by i = 8.243216% > 8%.

In the fractional accumulation with frequency *m (*o *m-fractionated*), if i is the effective annual rate, then

– the *accumulation factor* for the length $1/m$ is: $u_{1/\mathrm{m}} = 1+i_{1/\mathrm{m}}$;

– the *per period interest rate* for the length $1/m$ is:

$$i_{1/m} = (1 + i)^{1/m} - 1 \tag{3.26}$$

which is found from the equivalence relation between rates: $(1+i_{1/m})^m = 1+i$;

– the *per period interest intensity* for the duration $1/m$ is:

$$j(m) = m\ i_{1/m} \tag{3.26'}$$

and relation $i > j(m)$ can be deduced, if $m\text{-}1 \in \mathcal{N}$.

EXAMPLE 3.9.– We want to receive a return measure by the annual rate i = 6.45% with a prefixed use with monthly conversion. Then the monthly rate is $i_{1/12}$ = 0.522%, the corresponding intensity is $j(12)$ = 6.266802%/year and the monthly accumulation factor is $u_{1/12}$ = 1.00522.

Mixed accumulation with conversion m times per year

Using the assumption that the conversion is made at the end of each m^{th} of the solar year, if $f_1 < 1/m$ measures the interval between the investment and the first conversion and $f_2 < 1/m$ the interval between the last conversion and the end of the operation, by a generalization of (3.25) and using $t = f_1 + k/m + f_2$, we obtain:

$$M(t) = C(1 + j(m)\, f_1)\, (1 + j(m)/m)^k\, (1 + j(m)\, f_2) \tag{3.27}$$

EXAMPLE 3.10.– On 4 September 1996, Mr. Tizio withdraws €23,500 from a bank account ruled by a nominal 4-convertible rate = 4.65%/year, with mixed accumulation quarterly converted. The debt on 20 October 1999 is

$$M = 23500\cdot(1+0.0465\cdot 26/360)\cdot(1+0.0465/4)^{12}\cdot(1+0.0465\cdot 20/360) = €27{,}157.$$

Note: comparing this with the results in Example 3.7 using equal time and rate, the increase of the amount, which goes from €27,114 to €27,157 due to the more frequently interest conversion, will be noticed.

Equivalent rate and intensity in the fractional conversion

Two compound accumulation laws, the first with annual conversion at rate i and the second with m-fractional conversion at per period rate $i_{1/m}$, are called *equivalent* if they give the same annual return. This happens if i and $i_{1/m}$ satisfy (3.26); in this case such rates are said to be *equivalent.*

More generally, two compound accumulation laws, the first with m'-fractional conversion at rate $i_{1/m'}$ and the second with m''-fractional conversion at rate $i_{1/m''}$, are called, for the same reason, *equivalent* if

$$(1+i_{1/m'})^{m'} = (1+i_{1/m''})^{m''} \tag{3.28}$$

and then $i_{1/m'}$ and $i_{1/m''}$ are called *per period equivalent rates*.

An analogous definition for the intensities can be given. Due to (3.26') and (3.28), if

$$(1+\frac{j(m')}{m'})^{m'} = (1+\frac{j(m'')}{m''})^{m''} \tag{3.29}$$

is true, then *j(m')* and *j(m")* are *equivalent intensities.*

Exercise 3.11

Calculate the per period rates and intensities for the annual, semi-annual, four-monthly, quarterly, bimonthly, monthly, weekly, daily conversion frequencies in the compound regime at the annual rate of 5.27% and the quarterly rate of 1.36%, using Excel.

A. The given frequencies are: m = 1, 2, 3, 4, 6, 12, 52, 360. To obtain the solution we will use an Excel spreadsheet, which is particularly useful for calculating formulae with repeated structures (here varying m), using the “copy and paste” function. This is because in Excel the “copy” operation does not refer to the number in the cell but to the formula written in this cell, which works on the values written in other cells; besides, by “pasting” into another cell the formula is “translated”, i.e. it works on the cells corresponding by translation (unless the command $ is used). For example, if C6 includes a formula depending on the contents of the cells A9 and B10, by copying C6 and pasting in C9, the result is the value of the same formula applied on the contents of the cells A12 and B13: indeed, there is a three cell translation downwards. Consequently, changing data on the cells, all the results are instantaneously changed, which is very advantageous. This should be remembered for all exercises in this book that use Excel.

Using an Excel spreadsheet, using such techniques we will find the solutions based on (3.26), (3.26') and (3.28), (3.29) starting from the given rates 5.27% (annual) and 1.36% (quarterly). The following table is obtained.

CALCULATION OF EQUIVALENT RATES AND INTENSITIES

	Equivalent to rate i = 5.27%		Equivalent to rate $i_{1/4}$ = 1.36%	
m	$i_{1/m}$	*j(m)*	$i_{1/m}$	*j(m)*
1	5.270%	0.05270	5.552%	0.05552
2	2.601%	0.05202	2.738%	0.05477
3	1.727%	0.05180	1.817%	0.05452
4	1.292%	0.05169	1.360%	0.05440
6	0.860%	0.05158	0.905%	0.05428
12	0.429%	0.05147	0.451%	0.05416
52	0.099%	0.05138	0.104%	0.05406
360	0.014%	0.05136	0.015%	0.05404

Table 3.1. *Equivalent rates and intensities*

The Excel instructions are as follows. The first three rows are used for data and titles; D3: 0.0527; G3: 0.0136. The 4th row is empty. The 5th row has the column titles; from the 6th to 13th rows:

– column A (frequency): given frequency;

– column B: empty;

– column C (equivalent rates): C6:= (1+$D$3)^(1/A6)-1; copy C6, then paste on C7 to C13;

– column D (equivalent intensity) D6:= A6*C6; copy D6, then paste on D7 to D13;

– column E: empty;

– column F (equivalent rate): F6:= (1+$G$3)^(4/A6)-1; copy F6, then paste on F7 to F13;

– column G (equivalent intensity): G6:= A6*F6; copy D6, then paste on G7 to G13.

Note: rates are expressed in %; intensities are expressed in unitary form.

Effects of frequency variations

It is instructive to assess the effects on returns connected with a change of the conversion frequency, observing that:

a) if the intensity *j*, i.e. the flow of interest accruing divided by the updated principal, is fixed (constant in the time), the annual rate *i* that measures the return of the unitary principal after one year of investment with *m* equally spaced conversions is given by

$$i = f(j,m) = (1+\frac{j}{m})^m - 1 \tag{3.30}$$

which is a sequence increasing with *m*;

b) if the annual rate *i*, i.e. the return of a unitary principal after one year of investment with *m* equally spaced conversions, is fixed, the intensity *j* (constant in the time) is given by

$$j = g(i,m) = m[(1+i)^{1/m} - 1] \tag{3.31}$$

which is a sequence decreasing with *m*.

EXAMPLE 3.11

a) Let the intensity be j = 12% per year, i.e. it is established that within each interval between two subsequent conversions, the interest, which is still unprofitable, on the profitable sum *S(t)* is earned according to the flow 0.12·*S(t)*; it is then the product of such flow and the considered length Δ*t*. The interest earned after one year is an increasing function of the number *m* of conversions, each done after 1/*m* of a year, and is given by the product *Si*, where $i = f(j,m)$ is given, for the usual choices of *m*, by the values in the 3rd column of Table 3.2 below, obtained using (3.30).

b) Let the delayed annual interest be i = 12%; it is then established that, whatever the number *m* of conversions in one year, the intensity *j* (constant in the time) is such to assure at the end of the year of investment and interest return equal to 0.12·*C*, where *C* is the principal. With the increase of *m* the intensity $j = g(i,m)$ decreases and assumes, for the usual choices of *m*, the values in the 4th column in Table 3.2, obtained using (3.31).

The calculations are made on an Excel spreadsheet.

		Problem a) j = 0.12	*Problem b) i = 0.12*
Conversion frequency	*M*	*i, given j*	*j, given i*
Annual	1	0.120000	0.120000
Semi-annual	2	0.123600	0.116601
Four-monthly	3	0.124864	0.115496
Quarterly	4	0.125509	0.114949
Bimonthly	6	0.126162	0.114406
Monthly	12	0.126825	0.113866
Weekly	52	0.127341	0.113452
Daily	360	0.127474	0.113347

Table 3.2. *Correspondence between i and j*

The Excel instructions are as follows. The first three rows are used for data and titles; C3: 0.12; D3: 0.12. The 4th row is empty. The 5th row has column titles. From the 6th to 13th rows:

– column A: conversion frequency;

– column B (frequency): given frequency;

– column C (equiv. annual rat.) C6:= (1+C$3/B6)^B6-1; copy C6, then paste on C7 to C13;

– column D (equiv. intensity) D6:= B6*((1+D$3)^(1/B6)-1); copy D6, then paste on D7 to D13.

3.7.3. *The regime of continuously compound interest (CCI)*

We showed in section 3.7.1 that perfect synchrony of the supplies between the two contracting parties of a financial investment is obtained only with the *CCI* regime, which makes the accumulation with continuous conversion of interest that is accrued during the use of the capital[13]. The mathematical calculations have the difficulty of considering infinitesimal times, and it is necessary to use infinitesimal calculus. We will keep the hypothesis of per period rates and intensities constant in time.

We can consider two different ways to undertake the calculation:

1) the first is to assume the continuous conversion as the limit case of the fractional conversion when the frequency diverges (i.e. $m \to +\infty$);

2) the second, having general validity and also being suitable to describe the eventuality of time variable returns, consists of a direct approach to the formation of interest and their conversion, described with differential calculus. This is spontaneously related, in the case of constant in time returns, to the *exponential regime* described in section 2.6. We showed that the laws for such a regime, and only these, satisfy the properties of decomposability (and of strong decomposability, if we consider the couple of conjugate interest and discount laws) and uniformity in time.

The *first way* brings us to consider the limit of (3.30) and (3.31) with diverging m. By the limit of (3.30), given the instantaneous intensity (constant over time) of return, denoted by δ, we obtain the equivalent annual rate i, which is also the upper bound for $m = 1$, of the intensities $j(m)$ referred to the fractional conversion (according to the convexity of $e^{\delta t}$). By the limit of (3.31), given the annual rate i, we obtain the equivalent instantaneous intensity δ, lower bound for $m \to +\infty$ of the intensities $j(m)$. Using formulae

$$\begin{cases} i = \lim_{m\to\infty} f(\delta,m) = \lim_{m\to\infty}\left\{\left[1+\dfrac{\delta}{m}\right]^m - 1\right\} = e^{\delta} - 1 \\ \delta = \lim_{m\to\infty} g(i,m) = \lim_{m\to\infty} \dfrac{(1+i)^{1/m} - 1}{1/m} = ln(1+i) \end{cases} \qquad (3.30')$$

EXAMPLE 3.12.– By using the data in Table 3.2, given the constant intensity $j = 0.12$ and taking the limit $m \to +\infty$, it is calculated that in continuous accumulation $i = f(0.12,+\infty) = 0.1274969$ holds. Instead, using the effective annual rate $i = 0.12$, it is calculated that in continuous accumulation the instantaneous intensity $\delta = g(0.12,+\infty) = 0.1133287$ holds. By comparing these results with the last

13 In the hydraulic analogy of footnote 10, in continuous accumulation the second tank always has the input and output taps open, so that the “drops” of interest just formed go to the first tank and the second tank is almost always empty.

row of Table 3.2, it can be seen that the daily values (m = 360) are a good approximation of the continuous conversion's values.

An annual time horizon is not needed to define fractional and continuous accumulation. More generally, accumulating in the interval [0,T] using the intensity j, the amount in T corresponding to the principal C invested in 0 with equally spaced conversions in [0,T] is:

$$M(T) = C(1 + j\frac{T}{m})^m \tag{3.32}$$

and taking the limit for $m \to +\infty$, if the intensity $j(i,m)$ (varying with m so that i remains unchanged) converges to a real value indicated with δ, in CCI the following is obtained[14]

$$M(T) = \lim_{m\to\infty}\left\{C(1 + j(i,m)\frac{T}{m})^{m}\right\} = \tag{3.33}$$

$$= C \lim_{m\to\infty}\left\{(1+\frac{j(i,m)T}{m})^{\frac{m}{j(i,m)T}}\right\}^{j(i,m)T} = C\, e^{j(i,+\infty)\,T} = C\, e^{\delta T}$$

The *second way* formalizes the continuous conversion with constant rate. It follows from the following postulates:

– the *linearity*, that is, the proportionality between interests flow and the principal that generates them;

– the *circularity*, that is, the immediate and continuous transferring of earned interests to the profitable fund that generates them.

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14 This formulation of continuous accumulation, intended as a limit of the fractional accumulation, introduces the restriction of all equal subdivision intervals. Furthermore, in this limit we need the convergence of the intensity as a function of the fractioning. This last property exists in both cases examined in the table with varying m:

– when in the different fractioning situations the annual rate i (or a given per period rate $i_{1/m}$) is kept unchanged because in such a hypothesis the intensity, being decreasing, converges to $\delta = \ln(1+i) = \ln(1+i_{1/m})^m\}$;

– when the intensity does not change.

From these follows the equality between the amount's increment between t and $t+dt$, approximated by $dM(t)=M'(t)dt$, and the infinitesimal interest $\delta \cdot M(t) \cdot dt$. Then the simple differential equation (which is linear homogenous of the 1st order and with separable variables) is derived

$$M'(t) = \delta M(t) \tag{3.34}$$

for which the particular solution, relative to the condition $M(0) = C$, is

$$M(t) = C\, e^{\delta t},\ \forall t \in [0,T] \text{ [15]} \tag{3.35}$$

Equation (3.34) can be obtained with more details from the following considerations. Given the constant intensity $\delta > 0$, investing C at time 0 and without any interest conversion, at time T the interest is δC and if at that time the interest is added to the principal, the amount $M(t)$ becomes $C(1+\delta T)$. This SDI scheme satisfies linearity but not circularity, in the time interval $[0,T]$, where circularity instead implies that in the infinitesimal interval dt between times t and $t+dt$ in $[0,T]$, the amount is increased by the earned interest, expressed by $\delta \cdot M(t)dt + o(dt)$, where $o(dt)$ represents an infinitesimal error of order greater than dt. The following differential relation holds, $\forall t \in [0,T]$

$$M(t+dt) = M(t) + \delta M(t)dt + o(dt) \tag{3.34'}$$

which gives the amount, originated by the principal C invested in $t=0$ and without any other financial flow, as a function of t that is continuous and differentiable $\forall t > 0$. Taking the limit for $dt \to 0$ and taking into account that $\lim o(dt)/dt = 0$, (3.34) is obtained.

It is obvious that such a financial mechanism, based on linearity and circularity, realizes the *CCI regime with constant rate*, that, taking into account (2.50), is

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15 Considering, for the sake of simplicity, the unitary capital invested at time 0, the accumulated amount in t without previous conversion (then only due to linearity) is: $M_1(t) = 1+\delta t$, while with previous continuous conversion (then also due to circularity) it is

$$M_\infty(t) = e^{\delta t} = 1 + \delta t + \frac{\delta^2}{2} t^2 + \frac{\delta^3}{3!} t^3 + \ldots + \frac{\delta^n}{n!} t^n + \ldots > 1 + \delta t = M_1(t)$$

which shows that $M_\infty(t) > M_1(t)$ and also $M_1(t)$ is the linear approximation in $t=0$ of $M_\infty(t)$.

equivalent to the accumulation with the *exponential regime*[16]. A law of such a regime, and which is called an *exponential law*, applicable in the interval $[0,T]$, given the annual rate i or the per period rate $i_{1/m}$, is obtained from (3.35) using (see footnote 15)

$$\delta = \ln(1+i) = m \ln(1+i_{1/m}). \quad (3.31')$$

If the intensity δ is given, the following inverse formulae hold

$$i = e^{\delta} - 1 \quad ; \quad i_{1/m} = e^{\delta/m} - 1. \quad (3.31'')$$

Then, given the annual rate i, (3.35) can be written as

$$M(t) = C\,(1+i)^t,\ t \geq 0 \quad (3.35')$$

It follows from (3.35) and (3.35') that for the accumulation laws in the CCI regime[17]:

– the *accumulation factor* for the length $t = y\text{-}x > 0$ is

$$u_t = (1+i)^t = e^{\delta t} \quad (3.36)$$

– the *per period interest rate* for the length t is

$$i_t = (1+i)^t - 1 = e^{\delta t} - 1 \quad (3.37)$$

– the *per period interest intensity* for the length t is

$$j_t = i_t / t = [(1+i)^t - 1] / t = (e^{\delta t} - 1)/t \quad (3.38)$$

Exercise 3.12

Let us consider an investment of €4,550 in the CCI regime at the annual rate of 6.78% for 5 months and 18 days. Calculate the accumulation factor, the per period rate and the corresponding intensity, the instantaneous intensity and the earned interest at time t.

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16 Let us consider here the exponential regime in the formation of interests for problems of accumulation with constant rate (or with "flat structure"). For the analogous discount regime see section 3.8.

17 For these and other questions of financial mathematics, see S. Kellison (1991), Irwin; Poncet, Portait, Hayart (1993).

A. Adopting the bank year, this results in:

t = 5/12+18/360 = 0.46667; then:

$u_t = 1.0678^{0.466667} = 1.031087$; $i_t = 1.067^{0.466667} - 1 = 0.031087$;

j_t = 0.031087/0.466667 = 0.066615/year;

δ = ln 1.0678 = 0.0656/year; I_t = 4550·0.031087 = €141.45.

Exercise 3.13

Let us consider the investment of the previous exercise with the same interest rate but for a length of 2 years, 3 months and 7 days. Calculate the accumulation factor, the per period rate, the corresponding intensity and the interest earned at time t.

A. t = 2+3/12+7/360 = 2.269444 holds, and then:

$u_t = 1.0678^{2.269444} = 1.161023$; $i_t = 1.0678^{2.269444} - 1 = 0.161023$;

j_t = 1.161023/2.269444 = 0.070953; I_t = 4550·0.160123 = €732.65.

The problem of equivalent rate and intensities in the CCI regime is resolved by a generalization of (3.28) and (3.29), which is useable only if we consider natural numbers >1, since now we have to assume $t \in \Re^+$. Two *per period rates* for different periods t' and t'' are *equivalent* if, expressed as annual rates in the aforementioned regime, they give the same return in terms of rate i or instantaneous intensity δ. Two *per period intensities* are equivalent if they correspond to equivalent rates. In formulae, to have equivalence, the rates $i_{t'}$ and $i_{t''}$ must satisfy

$$(1+i_{t'})^{1/t'} = (1+i_{t''})^{1/t''} \; (= 1+i = e^{\delta}) \tag{3.39}$$

and the intensities $j_{t'}$ and $j_{t''}$ must satisfy

$$(1+ j_{t'}\, t')^{1/t'} = (1+j_{t''}\, t'')^{1/t''} \; (= 1+i = e^{\delta}) \tag{3.40}$$

Exercise 3.14

Let us consider an investment of €156,000 in the CCI regime for the length t' = (7m+24d) = 0.651620 year at the per period rate 0.0371. Calculate: 1) the corresponding intensity; 2) the rates and intensities equivalent to the previous ones, extending the investment for the length t'' = (1y+4m+17d)=1.380556 year; 3) the interest earned after one year of investment.

A. Using (3.39) and (3.40) the following is obtained:

1) $j_{t'}$ = 0.0371/0.651620 = 0.056935/year;

2) $i_{t''} = (1+i_{t'})^{t''/t'} - 1 = 1.0371^{2.118652} - 1 = 0.080235$; the intensity $j(t'')$ follows from (3.40) or (3.38), which leads to

$$j_{t''} = [\{1+_{t'}j_{t'}\}^{t''/t'}-1]/t'' = [\{1 + 0.056935\cdot 0.651620\}^{2.118652}-1]/1.380556$$
$$= 0.058118; \text{ or } j_{t''} = i_{t''}/t'' = 0.080235 / 1.380556 = 0.058118;$$

3) by inverting (3.37) the equivalent annual rate is obtained $i = 1.0371^{1/0.651620} - 1 = 0.057496$, and then the interest for one year of investment is

$$I_1 = 156{,}000\cdot 0.057496 = €8{,}969.38.$$

3.8. The regime of continuously compound discount (CCD)

We now consider the compound discount, only with regard to the *continuously compound discount (CCD)* (or *exponential*) regime which gives rise to a family of discount laws conjugated to those of CCI that can be specified by the instantaneous discount intensity θ. The function *C(t) = discount value of M for effect of an anticipation of length t* verifies the differential relation:

$$C(t+dt) = C(t) - \theta C(t)dt - o(dt) \tag{3.41}$$

(where $\theta C(t)dt$ is the *elementary discount* between t and $t+dt$) under the initial condition $C(0) = M$. Then $C(t)$ is expressed by

$$C(t) = M\, e^{-\theta t} \tag{3.42}$$

Recalling (3.35), it is obvious that the law of exponential discount in (3.38) with parameter θ is conjugated to the law of exponential accumulation with parameter δ if and only if θ = δ.

Working with a CCD law characterized by the intensity θ on *annual interval* ($t = 1$) or *fraction of year* ($t = 1/m$), it follows from (3.42) that the *annual discount factor v*, the *per period discount factor* $v_{1/m}$, the *annual discount rate d* and the *per period discount rate* $d_{1/m}$ for time $1/m$ are given respectively by:

$$v = C(1)/M = e^{-\theta}\ ;\ v_{1/m} = C(1/m)/M = e^{-\theta/m} \tag{3.43}$$

$$d = \{M - C(1)\}/M = 1 - v = 1 - e^{-\theta} \tag{3.44}$$

$$d_{1/m} = \{M - C(1/m)\}/M = 1 - e^{-\theta/m} \tag{3.45}$$

resulting in the following equivalence relations on discount rates for different frequencies

$$(1 - d_{1/m'})^{m'} = (1 - d_{1/m})^{m} = 1 - d \tag{3.46}$$

In addition, the definition of *per period discount intensity* relative to the frequency m (also called *nominal discount rate convertible m times a year*) is expressed by

$$\rho(m) = m\, d_{1/m} = m\,(1 - e^{-\theta/m}) \tag{3.47}$$

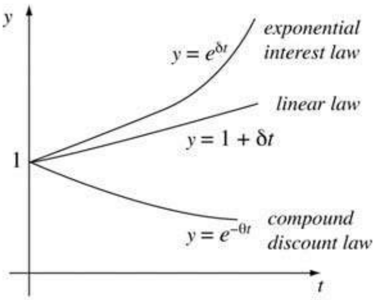

Figure 3.4. *Interest and discount exponential law*

EXAMPLE 3.13.– Considering the CCD law with $\theta = 0.0689$, we obtain for the factor and the rate of annual discount the values $v = e^{-0.0689} = 0.93342$ and $d = 1 - v = 0.06658$, while for the quarterly discount length (m=4) we obtain for the per period factor, the per period rate and the corresponding intensities the following values:

$v_{1/4} = e^{-0.0689/4} = 0.982923$; $d_{1/4} = 1 - v_{1/4} = 0.017077$; $\rho(4) = 4\, d_{1/4} = 0.06831$.

If $m' = 6$ (= bimonthly period), for (3.46) the equivalent discount rate is $d_{1/6} = 1-(1-d_{1/4})^{4/6} = 0.011418$.

Comparing (3.45) with (3.26), the rates $i_{1/m}$ and $d_{1/m}$ come from m-fractional and conjugated compound laws if the following relation holds:

$$(1 + i_{1/m})(1 - d_{1/m}) = 1 \tag{3.48}$$

and, taking into account (3.26') and (3.47), the intensities for conjugated laws satisfy the relation

$$(1 + j(m)/m)(1 - \rho(m)/m) = 1 \tag{3.48'}$$

Exercise 3.15

Using the CCD law with instantaneous intensity θ = 0.0523, calculate the rates and the per period intensities, equivalent to each other, of such a law for the usual frequencies. Also calculate the rates and the per period intensities of interest for the same frequencies, based on the instantaneous intensity δ = θ or δ = 0.0473 ≠ θ.

A. Using Excel, the rates $d_{1/m}$ and the intensities $\rho(m)$ for changing m are obtained using (3.45) and (3.47). Furthermore, if δ = θ = 0.0523, the CCI law is conjugated to the CCD law; so the rates $i_{1/m}$ and the intensities $j(m)$ are obtained using (3.48) and (3.49). The following table is obtained.

CALCULATION OF EQUIVALENT PER PERIOD RATES AND INTENSITIES
with conjugated CCD and CCI laws

	intensity θ =	intensity δ =	0.0523	
m	$D_{1/m}$	. θ*(m)*	$i_{1/m}$	*J(m)*
1	5.096%	0.05096	5.369%	0.05369
2	2.581%	0.05162	2.649%	0.05299
3	1. 728%	0.05185	1.759%	0.05276
4	1.299%	0.05196	1.316%	0.05264
6	0.868%	0.05207	0.875%	0.05253
12	0.435%	0.05219	0.437%	0.05241
52	0.101%	0.05227	0.101%	0.05233
360	0.015%	0.05230	0.015%	0.05230

Table 3.3. *Equivalent per period rates and intensities*

The Excel instructions are as follows. Rows 1, 2, 4, 5 are for data and titles; F4: 0.0523. The 4th row is empty. From the 6th to 13th rows:

– column A (frequency): insert the given frequencies;

– column B: empty;

– column C (per period disc. rate): C6:= 1-EXP(-$F$4/A6);

– column D (per period disc. intensity): D6:= A6*C6; copy D6, then paste on D7 to D13;

– column E: empty;

– column F (per period interest rate): F6:= 1/(1-C6)-1; copy F6, then paste on F7 to F13;

– column G (per period interest intensity): G6:= A6*(1/(1-D6/A6)-1); copy G6, then paste on G7 to G13.

The convergence of the per period intensities to $\delta = \theta = 0.0523$ is verified.

If instead $\delta = 0.0473$, the laws are not conjugated and the calculation of the interest rates and intensities proceeds autonomously on the basis of (3.31') and (3.38) with $t = 1/m$. We then obtain the following table.

CALCULATION OF EQUIVALENT PER PERIOD RATES AND INTENSITIES
with unconjugated CCD and CCI laws

	intensity θ =	0.0523	intensity δ =	0.0473
m	*d.1/m*	Q*(m)*	*i.1/m*	*J(m)*
1	5.096%	0.05096	4.844%	0.04844
2	2.581%	0.05162	2.393%	0.04786
3	1.728%	0.05185	1.589%	0.04767
4	1.299%	0.05196	1.190%	0.04758
6	0.868%	0.05207	0.791%	0.04749
12	0.435%	0.05219	0.395%	0.04739
52	0.101%	0.05227	0.091%	0.04732
360	0. 015%	0.05230	0.013%	0.04730

Table 3.4. *Equivalent per period rates and intensities*

The Excel instructions are as follows. Rows 1, 2, 4, 5 are for data and titles; D4: 0.0523; G4: 0.0473; the 4^{th} row is empty; from the 6^{th} to 13^{th} rows:

– column A (frequency): insert the given frequencies;

– column B: empty;

– column C (per period disc. rate): C6:= 1-EXP(-$D$4/A6);

– column D (per period disc. intensity): D6:= A6*C6; copy D6, then paste on D7 to D13;

– column E: empty;

– column F (per period interest rate): F6:= EXP($G$4/A6)-1; copy F6, then paste on F7 to F13;

– column G (per period interest intensity): G6:= A6*F6; copy G6, then paste on G7 to G13.

This verifies the convergence of the per period intensities to the respective instantaneous intensities with frequency divergence.

In general, with diverging *m*, (3.47) converges to the instantaneous intensity θ. Indeed, using $h = -1/m$

$$\lim_{m\to\infty} \rho(m) = \lim_{h\to 0} (e^{\theta h}-1)/h = \theta \tag{3.49}$$

Working with a CCD law whose instantaneous intensity is θ, on *any discount length t* >0 due to (3.42) the *discount factor*, the *per period discount rate* and the *per period discount intensity* for the length *t* are given respectively by

$$v_t = e^{-\theta t} \tag{3.50}$$

$$d_t = 1 - e^{-\theta t} \tag{3.51}$$

$$\rho_t = d_t / t = (1 - e^{-\theta t})/t \tag{3.52}$$

EXAMPLE 3.14.– Using the same discount law as in Example 3.13 and applying t = (2y+7m+21d) = 2.641667, the following values for (3.50), (3.51) and (3.52) are obtained:

$$v_t = 0.83359229;\ d_t = 0.16640771;\ \rho_t = 0.06299345$$

The comparison between (3.50) and (3.36) shows that if $\theta = \delta = \ln(1+i)$, u_t and v_t are reciprocal, where the corresponding CCI and CCD laws are conjugated.

3.9. Complements and exercises on compound regimes

Complement 1: graphical interpretation

Let us recall that the exponential functions differ from their reciprocal functions only by the sign of the exponent: $1/e^{\delta\tau} = e^{-\delta\tau} = e^{\delta(-\tau)}$. Therefore, it can be concluded that the same function $e^{\delta\tau}$ represents, depending on the sign of τ, compound accumulation or discount, if we consider the following durations:

– in the first case a positive duration $\tau > 0$ between the beginning and the end of accumulation;

– in the second case a negative duration proceeding backwards from the maturity, taken as origin, until time $\tau < 0$ where the discount is carried out.

This enables us to represent in only one graph $f(t)= e^{\delta t}$, $\forall$t, shown in Figure 3.5, the typical quantities of the exponential regime, choosing an intensity $\delta>0$ which represents the interest intensity for the accumulation law and the discount intensity for the discounting law.

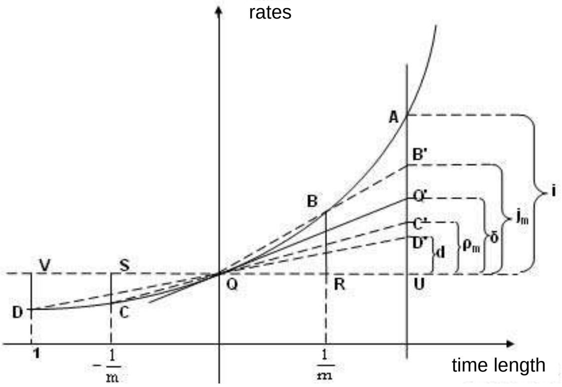

Figure 3.5. *Rates and intensities in the exponential law*

Interpretation of Figure 3.5

Let us consider in Figure 3.5 the following typical points of the graph of the function $f(t)$ = $e^{\delta t}$ identified by the Cartesian coordinates on the plain O*tf*:

$$A = (1,e^{\delta});\ B = (1/m,e^{\delta/m});\ Q = (0,1);\ C = (-1/m,e^{-\delta/m});\ D = (-1,e^{-\delta})$$

where m is the conversion frequency. The points B', C', D' are intersections with t=1 of the secants of the exponential f = $e^{\delta t}$ respectively for the fixed point Q and the varying points B, C, D; furthermore, point Q' is the intersection in t=1 of the tangent of the curve in Q (= limit line of the secants). The point U, R, S, V on the horizontal $f=1$ have the same abscissa as A, B, C, D.

Let us observe that because of the proportionality between catheti of similar triangles QRB and QUB': $\overline{UB'} = \overline{RB}/\overline{QR}$ = slope of the secant QB. Using a similar argument: $\overline{UC'} = \overline{SC}/\overline{QS}$ = slope of the secant QC. Because $\overline{QU}=1$, then UQ' = slope of the line QQ' tangent in Q to $e^{\delta t}$ as well.

Because $e^{\delta} = 1+i$, $e^{\delta/m} = 1+i_{1/}$, $e^{-\delta/m} = 1-d_{1/m}$, $e^{-\delta} = 1-d$ and also $j(m) = m{\cdot}i_{1/m}$, $\rho(m) = m{\cdot}d_{1/m}$, the following graphical interpretation can be obtained:

– ordinate of D = v = discount factor for one year;

– ordinate of C = $v^{1/m}$ = discount factor for 1/m of one year;

– ordinate of B = $u^{1/m}$ = accumulation factor for 1/m of one year;

– ordinate of A = u = accumulation factor for one year;

– $\overline{VD} = \overline{UD'} = d = \rho(1)$ = annual discount rate = discount intensity on one year;

– $\overline{RB} = d_{1/m}$ = discount rate per period for 1/m of one year;

– $\overline{SC} = i_{1/m}$ = interest rate per period for 1/m of one year;

– $\overline{UA} = i = j(1)$ = annual interest rate = interest intensity on one year base;

– $\overline{UB'} = j(m)$ = interest intensity per period on 1/m of one year;

– $\overline{UQ'} = \delta$ = instantaneous interest intensity;

– $\overline{UC'} = \rho(m)$ = discount intensity per period on 1/m of one year.

It is clear that rates and intensities relative to different periods, taken from the same function $e^{\delta t}$, are equivalent.

Using the same graph as in Figure 3.5, the relations between the fundamental quantities u, v, i, d (which, referring to one year, are valid for all uniform regimes considered in this chapter) can be considered, as well as the relations between the fundamental quantities and the instantaneous intensity δ. Such relations are summarized in the following table, where each quantity given in the 1st column is expressed as a function of the quantities given in the 1st row.

	u	**v**	**i**	**d**	**δ**
u	u	$\frac{1}{v}$	$1+i$	$\frac{1}{1-d}$	e^{δ}
v	$\frac{1}{u}$	v	$\frac{1}{1+i}$	$1-d$	$e^{-\delta}$
i	$u-1$	$\frac{1-v}{v}$	i	$\frac{d}{1-d}$	$e^{\delta}-1$
d	$\frac{u-1}{u}$	$1-v$	$\frac{i}{1+i}$	d	$1-e^{-\delta}$
δ	$\ln u$	$-\ln v$	$\ln(1+i)$	$-\ln(1-d)$	δ

(3.53)

Table 3.5. *Transformation formulae between rates or intensities*

Complement 2: average length and average rate in the compound regime

By applying the same considerations as in sections 2.5.3 and 2.5.4 to the compound regime, it can be easily verified that:

– using a CCI law and the accumulation factor $u_t = (1+i)^t$, the *average length* $\hat{t}$ (equal to the average term if the investment starts at 0) is given by the exponential mean with base (1+*i*) of the length t_h of the investment on the principal C_h. Then:

$$(1+i)^{\hat{t}} = \frac{\sum_{h=1}^{n} C_h (1+i)^{t_h}}{\sum_{h=1}^{n} C_h} \tag{3.54}$$

In the same way, using a CCD law and the discount factor $v_t = (1+i)^{-t}$, the *average length* $\hat{t}$ is given by the exponential mean with base $(1+i)^{-1}$ of the length t_h of the discount on the terminal amount M_h. Then:

$$(1+i)^{-\hat{t}} = \frac{\sum_{h=1}^{n} M_h (1+i)^{-t_h}}{\sum_{h=1}^{n} M_h} \tag{3.55}$$

– using a CCI law, the *average rate* $\hat{i}$ relative to the investment of principal C_h for the same length t made with rate i_h is the mean of powers with exponent t defined by

$$\left(1+\hat{i}\right)^t = \frac{\sum_{h=1}^{n} C_h \left(1+i_h\right)^t}{\sum_{h=1}^{n} C_h} \tag{3.56}$$

In the same way, using a CCD law, the *average rate* $\hat{i}$ relative to the discount on terminal value M_h for the same length t made at rate i_h is the mean of powers with exponent -t defined by

$$\left(1+\hat{i}\right)^{-t} = \frac{\sum_{h=1}^{n} M_h \left(1+i_h\right)^{-t}}{\sum_{h=1}^{n} M_h} \tag{3.57}$$

Complement 3: plurality of accounts and problems of averaging

Let us consider the following application which implies an averaging problem. A company has to operate financially through a plurality of accounts, all ruled by a compound regime, which is decomposable, but with different rates. Let u_h be the annual accumulation factor on the principal $C_h > 0$ invested at time 0 in the h^{th} account (h = 1, ..., n).

We are interested in valuing the characteristics of this accumulation regime connected to the *total financial management* of the n account, considering only the effect of such initial investments. So the accumulation factor for the 1st year is $m(0,1) = \sum_h C_h u_h / \sum_h C_h = \hat{u}$ = weighted arithmetic mean of the single factors (= *first moment* of the distribution $\{u_h, C_h\}$); for two years of consecutive investment the accumulation factor is: $m(0,2) = \sum_h C_h u_h^2 / \sum_h C_h$ = mean of squared u_h (= *second moment* of the distribution $\{u_h, C_h\}$).

The decomposability valid on each account is maintained at a global level as long as, supposing for example an interruption after one year, further accumulation for the 2nd year of the obtained amounts $C_h u_h$ is made $\forall h$ with the same factor u_h valid in the 1st year. The following is obtained indeed:

$$m(1,2) = \frac{\sum_h (C_h u_h) u_h}{\sum_h C_h u_h} = \frac{\sum_h C_h u_h^2}{\sum_h C_h u_h} = \frac{m(0,2)}{m(0,1)} \tag{3.58}$$

However, it can be observed that:

– the values in (3.58) are given by the anti-harmonic mean of the factors u_h, which is not associative[18];

– the total amount = $\sum_h C_h u_h$ at time 1 of the principal $\sum_h C_h$ can be obtained by also applying the mean annual rate $\hat{u}$ to each account, but if the rate $\hat{u}$ is also applied to each account in the 2nd year, after the interruption, we would obtain a lower total amount and the global process would not be separable. The result of such a hypothesis is that:

$$\mu_1 = m(0,2) / m(0,1) = \left(\sum_h C_h u_h^2\right) / \left(\hat{u} \sum_h C_h\right)$$

$$\mu_2 = m(1,2) = \sum_h \left(C_h u_h\right) \hat{u} / \sum_h C_h u_h = \hat{u}$$

Putting $\mu_3 = \left(\sum_h C_h u_h\right) \sum_h C_h > 0$, we obtain

$$(\mu_1 - \mu_2)\, \mu_3 = \sum_{h<k} C_h C_k (u_h - u_k)^2 > 0$$

and then $\mu_1 > \mu_2$. Thus, the statement is proved.

The problem is more complicated if some of the amounts are credits and some are debits, without the possibility of compensation.

Such simple observations should make the financial operator consider the delicacy of such problems and the attention needed in choices when averaged values are used.

Exercises on equivalent rates and intensities

It is convenient to stress that the consideration of a rate per period for 1/*m* of a year does not have meaning in annual conversion; it only has meaning in m-fractional conversion or *m'*-fractional, with *m'* multiple of *m*, or in an exponential

18 Generalizing this conclusion, we can observe that a feature of the compound regime is the fact that the continuing annual accumulation factor for the k^{th} year is the anti-harmonic mean of order *k* given by $\sum_h C_h u_h^h / \sum_h C_h u_h^{h-1}$; see: Caliri (1981).

regime ($m \to +\infty$). In the latter case the interest rate can be considered for any period t, expressed by $e^{\delta t}$-1.

Exercise 3.16

Firm Y receives from Bank X a short-term loan with 7.60% nominal annual rate with quarterly conversion and uses it in an operation with monthly income. Calculate the minimum monthly rate of return necessary to assure a positive spread of 2% on the cost rate in terms of effective annual rates.

A. The parameter 0.076 = nominal annual rate 4-convertible = $j(4)$, is an intensity, referred to the quarterly conversion. It corresponds to effective annual rate $i = (1+j(4))^4-1 = 0.078194$. Therefore, the minimum annual rate of return is: $i' =$ 0.098194, to which corresponds the monthly rate $(1+i')^{1/12} - 1 = 0.007836 =$ 0.7836%.

Exercise 3.17

For the loan of the principal C = €250,000, there will be delayed bimonthly interest payments of €2,900, until the time of repayment in one transaction.

Calculate the amount of per period equivalent interest payments:

a) in the case of monthly advance payments;

b) in the case of semi-annual delayed payments;

c) in the case of quarterly advance payments.

A. Having established the final repayment of the total loan, the installments paid by the debtor are pure interest. Furthermore, the equivalent installments have to be calculated using the same DCI law with the monthly conversion (monthly because 12 is the least common multiple of the frequencies considered here).

So, because of the data, the accumulation law used here gives rise to a value for the bimonthly interest rate equal to

$$i_{1/6} = 2{,}900/250{,}000 = 0.0116 = 1.16\%$$

Recalling (3.28), (3.46) and (3.48):

$$\text{a)}\ d_{1/12} = \frac{i_{1/12}}{1+i_{1/12}} = \frac{(1+i_{1/6})^{0.5}-1}{(1+i_{1/6})^{0.5}} = 0.005750$$

therefore the equivalent advance monthly installment is: $C\,d_{1/12}$ = €1,437.

b) $i_{1/2} = (1 + i_{1/6})^3 - 1 = 0.035205$

therefore the equivalent delayed semi-annual installment is: $C\,i_{1/2}$ = €8,801.

$$\text{c)}\ d_{1/4} = \frac{i_{1/4}}{1+i_{1/4}} = \frac{(1+i_{1/6})^{1.5}-1}{(1+i_{1/6})^{1.5}} = 0.017151$$

therefore the equivalent advance quarterly installment is: $C\,d_{1/4}$= €4,288.

Exercise 3.18

1) For an investment of €10,000 in compound regime at the annual effective rate of 5%, let us compare the amount after 5 years and 7 months in the three following options:

a) with CCI law;

b) with mixed law with quarterly conversion;

c) with mixed law with annual conversion.

For b) and c) use the assumption that the investment is made at one prefixed time of conversion (for example on 1 January).

A. In case a), apply (3.35'), use C = 10,000; i = 0.05; t = 5+7/12 = 5.583333; then:

$$M_a = 10{,}000 \cdot (1.05)^{5.583333} = €13{,}131.27.$$

In case b), apply (3.27) with $f_1 = 0$; $f_2 = 1/12$; $m = 4$; $k = 22$; $j(4) = 4\,(1.10^{1/4} - 1) = 0.040989$; C = 10,000; then:

$$M_b = 10{,}000 \cdot (1+0.040989/4)^{22} \cdot (1+0.040989 \cdot 0.083333) = €13{,}131.50.$$

In case c), apply (3.25) with $f_1 = 0$; $n = 5$; $f_2 = 7/12$; C = 10,000; then:

$$M_c = 10{,}000 \cdot (1.05)^5 \cdot (1+0.05 \cdot 0.583333) = €13{,}135.06.$$

The amounts are in increasing order, given that they follow from the same effective rates. In addition, M_b is very close to M_a.

2) Make the comparison for the amounts made in 1 but for a length of 5 years.

A. As will be shown in section 3.10, for integer length the three amounts are the same. For 5 years this gives: $M_a = M_b = M_c$ = 12762.82.

3) Make the comparison as in 1, but calculating for 5 years and 7 months with a common intensity j = 0.05 for any frequency of conversion.

A. In such a case, introducing j both in the compound law for integer year and in the linear law for fractions of a year, we obtain:

M_a = 10,000 $e^{0.055 \cdot 583333}$ = €13,220.27

M_b = 10,000 $(1+0.05/4)^{22}$ (1+0.05·0.083333) = €13,197.64

M_c = 10,000 $(1+0.05)^5$ (1+0.05·0.583333) = €13,135.06

The value M_c coincides with that in 1 because numerically $i = j(1)$. The amounts are now in decreasing order with the decreasing number of conversions ($M_a > M_c$ because $e^{\delta\tau} > 1+\delta$ t).

4) Make the comparison as in 1), but for 5 years as in 2).

A. Obviously the equality between the amounts is lost and then:

M_a = €12,840.25; M_b = €12,820.37; M_c = €13,762.82.

Exercise 3.19

Consider the same problem as in Exercise 3.18, 1), using the same data, but removing the assumption that the investment starts at the conversion dates, but instead starts 12 days in advance.

A. Using the bank year (= 12 months of 30 days each), results in:

– case a), no changes because the exponential law depends only on the total length, which has not changed; therefore, M_a = €13,131.27;

– cases b) and c) concern mixed law, then not a uniform law, and the result changes.

In case b), using in (3.27): f_1 = 0.03333 = (12 d); k = 22; f_2 = 0.05 (= 18 d); $j(4)$ = 0.049089; C = €10,000, the following is obtained:

M_b =10,000 (1+1.049089·0.03333)·$(1+0.049089/4)^{22}$ (1+0.049086·0.05) = €13,131.55.

In case c), putting in (3.25): f_1 = 0.033333; f_2 = 7/12 - 12/360 = 0.55; n = 5; i = 0.05; C = 10,000, the following is obtained

$$M_c = 10{,}000\ (1+1.05\cdot 0.03333)\cdot(1+0.05)^5\cdot(1+0.05\cdot 0.55) = €13{,}135.65.$$

If with the law assumed in case b), used in banks on current accounts, the fractions f_1 and f_2 are calculated relating the effective numbers of day to the bank year, i.e. 360, and can assume values greater than 1/4 (so that from 1 July to 29 September inclusive, there are 91 days, resulting in 91/360 = 0.252778>1/4).

Exercise 3.20

In Exercise 3.19 we verified that, with the same interest and length, in mixed accumulation the result changes according to the placement of the investment interval with respect to the conversion interval. Calculate the values that, using the same data, maximize the amount.

A. Considering for the sake of simplicity case c), we have to work on variables f_1 and f_2 such that $f_1+f_2 = t\text{-}n$ = constant = H and maintaining the number n+2 of conversions. Using $f_1 = x$, $f_2 = H\text{-}x$, with the data of Exercise 3.17 it is necessary to maximize the accumulation factor

$$g(x) = M(t)/C = (1 + 0.05x)\cdot 1.05^5\cdot[1 + 0.05(H\text{-}x)];$$

its graph is a concave downward parabola, thus having only one maximum point where the first derivative is zero. It is $g'(x) = 0$ for $x = H/2$, i.e. when $f_1 = f_2$.

In conclusion, if the length and frequency (annual, but this also holds for the fractional case, as it is easy to verify) are given, it is convenient for the creditor that the interval of investment is positioned symmetrically with respect to the conversion intervals.

EXAMPLE 3.15.– Given an investment for 3 years and 6 months between 2005 and 2009 at an annual rate of 5.50%, with conversion at the end of the calendar year, taking into account that the beginning cannot be before 1 July 2005 and the term cannot be after 3 June 2009, we obtain the maximum accumulation factor, equal to 1.206755, when the investment begins on 1 October 2005 and ends on 31 March 2009. Indicating by x the number of months in 2005 and by y = 6-x the number of months in 2009, by varying x with the respect of the given constraints, we obtain the following results which gives the order of magnitude of the variations.

Investment intervals	*x*	*y*	*Accumulation factor g(x)*
01/07/05 – 31/12/08	6	0	1.206533
01/08/05 – 31/01/09	5	1	1.206656
01/09/05 – 28/02/09	4	2	1.206730
01/10/05 – 31/03/09	3	3	1.206755
01/11/05 – 30/04/09	2	4	1.206730
01/12/05 – 30/05/09	1	5	1.206656
01/01/06 – 30/06/09	0	6	1.206533

Table 3.6. *Comparison among accumulation factors*

3.10. Comparison of laws of different regimes

After collecting the results of previous section we can make a comparison between the amounts obtainable with different uniform accumulation regimes already considered or between the present values connected with different uniform discount regimes.

We will consider in this section:

a) *in accumulation*, the comparison among simple, delayed or advance, and continuously compound interest laws;

b) *in discount*, the comparison among rational, simple and continuously compound discount laws.

The result of such a comparison depends on the functional form of the exchange factors but also on the return parameters (rates or intensities) used for the single laws.

When referring to the different *accumulation regimes*, if we only consider a comparison in the assumption of equal i, i.e. among:

– an SDI law with annual rate i;

– a CCI law with the same annual rate i;

– an SAI law with annual rate $d = i/(1+i)$;

we can conclude straight away that:

1) the three SDI, CCI and SAI laws give rise to the same return of interest after one year of investment, i.e. the *indifference length* is 1;

2) indicating here with $\succ$ the preference among laws

$$(\text{SDI}) \succ (\text{CCI}) \succ (\text{SAI}), \text{ if } t < 1,$$

$$(\text{SAI}) \succ (\text{CCI}) \quad (\text{SDI}), \text{ if } t > 1.$$

Regarding comparison among *discount regimes*, it is enough to observe that the RD, CCD and SD regimes give rise to conjugated laws, respectively, to SDI, CCI and SAI. Then it is enough to consider the reciprocal factors and repeat all reasoning, to conclude, when comparing the following:

– an RD law with annual rate i;

– a CCD law with the same annual rate;

– an SD law with annual rate $d = i/(1+i)$;

that

1) the three RD, CCD and SD laws give rise to the same discount return after one year of anticipation, i.e. the *indifference length* is 1;

2) the preference among laws, indicated here by ↾ , is

$$(\text{SD}) \succ (\text{CCD}) \succ (\text{RD}), \text{ if } t < 1,$$

$$(\text{RD}) \succ (\text{CCD}) \succ (\text{SD}), \text{ if } t > 1.$$

Graphical interpretation

Figure 3.6 shows the comparison among interest laws: (SDI)⟶ line (a), (CCI)⟶ line (b), (SAI)⟶ line (c), when the delayed interest rates coincide in the different law and the indifferent length is 1. The comparison among discount conjugate law (RD) ⟶line(a'), (CCD)⟶line(b'), (SD)⟶line(c'), with the same conditions and indifferent length, is also shown.

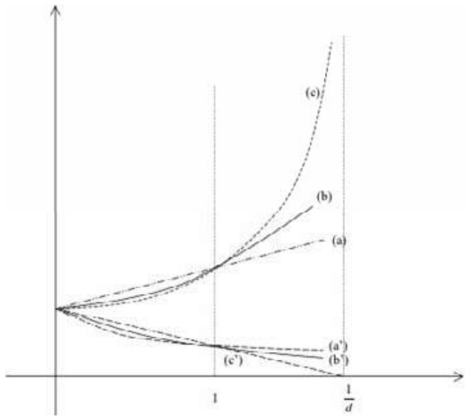

Figure 3.6. *Comparisons among interest and discount laws*

Let us now solve the *problem of comparing* the various regimes two by two, when different rates are applied to the laws of accumulation or discount. In this way we can also find the *indifference lengths* which depend on the couples of the chosen rates.

With reference to *interest laws*, the following results are obtained.

A1) *Comparison between SDI and CCI laws*

Let i_0 be the annual rate of an SDI law and i the annual effective rates for a CCI law. With reference to the accumulation factors, the principal and the amount being proportional, the returns coincide in both laws if the length t satisfies the relation

$$1 + i_0 t = (1 + i)^t \tag{3.59}$$

We will not consider the solution $t = 0$, because we are interested only in a positive solution t':

– if $i_0 > \delta = \ln(1+i)$, such a solution exists and is unique, given the upward concavity of $(1+i)^t$. The calculation of indifference length t' must be done numerically. If i and i_0 satisfy (3.59), $1+i_0 t > (1+i)^t$ if $t<t'$ holds, while $1+i_0 t < (1+i)^t$ if $t>t'$. Therefore, the compound law is preferable for the investor only for a length greater than the indifference length, which is

$$t' = 1/m \text{ if } i_0 = j(m);\ t' = 1 \text{ if } i_0 = i.$$

– if $i_0 \leq \delta$, there is no indifference length and the compound law is always preferable.

Exercise 3.21

Given the (SDI) law with an annual rate of i_0 = 0.061 and the (CCI) law with an annual rate of i = 0.062, calculate the indifference length using the methods described in this section.

A. Given the annual rate i_0 = 0.061 > ln(1+i) = 0.060154, there exists the indifference length t' > 0. We have t' = 1 if $i_0 = i$; but being $i_0 < i$, $t' < 1$ follows. Finally: $0 < t' < 1$ and the compound factor prevails if $t > t'$.

Indicating with $\zeta(t) = (1+i)^t - (1+i_0 t)$ the spread between the factors (where by definition $\zeta(t')=0$) is $\zeta(1) = i - i_0$ and with the given rates: $\zeta(1)$ = 0.001. Let us calculate in the interval (0,1) a time t such that $\zeta(t) < 0$. With decreasing t we have for example: $\zeta(0,4)$ = -0.000047. Proceeding initially with the *dichotomic method* (see section 4.5.3) between t=1 and t=0.40, we obtain: $\zeta(0.70)$ = 0.000307; $\zeta(0.55)$ = 0.000088; etc. The convergence is slow.

Let us proceed with the *secant method* (see section 4.5.4), with upper bound t = 0.55 fixed and increasing lower bound from t = 0.40.

1st step: linear interpolation between t = 0.40 and t = 0.55:

$$\frac{t-0.40}{0.55-0.40} = \frac{0-\zeta(0.40)}{\zeta(0.55)-\zeta(0.40)} = \frac{47}{88+47} = 0.348148$$

then t = 0.40 + 0.15· 0.348148 = 0.452222; $\zeta(t)$ = -0.000009.

2nd step: linear interpolation between t = 0.452222 and t = 0.55:

$$\frac{t-0.452222}{0.55-0.452222} = \frac{0-\zeta(0.452222)}{\zeta(0.55)-\zeta(0.452222)} = \frac{9}{88+9} = 0.092784$$

then t = 0.452222 + 0.097778· 0,092784 = 0.461294; $\zeta(t)$ = -0.000002.

3rd step: linear interpolation between t = 0.461294 and t = 0.55:

$$\frac{t-0.461294}{0.55-0.461294} = \frac{0-\zeta(0.461294)}{\zeta(0.55)-\zeta(0.461294)} = \frac{2}{88+2} = 0.022222$$

then t = 0.461294 + 0.088706· 0.022222 = 0.463265; $\zeta(t)$ = -0.000000035.

Let us stop the iterative process, because time 0.463265 = (5m+17d) is a good estimation (approximated by defect) of the indifference length t', implying a spread ζ of almost zero.

A2) *Comparison between SDI and SAI laws*

Let i be the annual rate of an SDI law and d the annual discount rate of an SAI law. We have coincidence of returns (for length $t < 1/d$) if

$$1 + it = (1 - d\,t)^{-1} \tag{3.60}$$

and we have the only positive solution $t' = (i_0-d)/i_0 d$ if and only if $i_0>d$. In particular $t' = 1$ if $i_0 = d/(1-d)$.

Due to the sign of concavity $(1 - dt)^{-1}$, the SDI law is convenient for the investor if $t<t'$, but the SAI law is convenient if $t>t'$.

EXAMPLE 3.16.– Comparing an SDI law with an annual rate i_0 = 4.70% with an SAI law with an annual advance rate d = 4.30%, the indifference length is given by:

$$t' = (0.047\text{-}0.043)/(0.047\cdot 0.043) = 1.979218 = 1\text{y}+11\text{m}+23\text{d}.$$

Using instead the corresponding rate d = 0.047/1.047 = 4.489% we obtain $t' = 1$.

A3) *Comparison between SAI and CCI laws*

Let d be the annual discount rate of a SAI law and i the effective annual rate of a CCI law. The returns are the same if the length satisfies the relation

$$(1 - d\,t)^{-1} = (1+i)^t \quad , \quad t < 1/d \tag{3.61}$$

For this comparison the calculation of indifference length t' must be performed numerically. We have a solution $t'>0$ (which can be shown to be unique) to the problem of equivalent length if and only if $d<\delta$.[19] In such a case, if $t<t'$ the CCI law is convenient for the investor; if $t>t'$, then the SAI laws are convenient. If instead $d>\delta$, the SAI law is always convenient for the investor.

19 This is because the curves $(1-dt)^{-1}$ and $(1+i)^t$ are both convex and have right derivatives in t=0 equal respectively to d and δ.

Exercise 3.22

Given the law (CCI) at the annual rate i = 0.062 and the law (SAI) at the annual delayed rate d = 0.059, calculate the indifference length using the method described in section 4.5.

A. Given that $d = 0.059 < \delta = \ln(1+i) = 0.060154$, there exists the indifference length $t' > 0$. To calculate this, we proceed as in Exercise 3.21, where the CCI and SDI laws are compared. Furthermore, with length t=1 the SAI law is convenient, because the following is obtained for the accumulation factors: $1/(1-d) = 1.062699>1.062000 = 1+i$. Then: $0<t'<1$ and the simple advance factor prevails if $t > t'$.

Indicating with $\xi(t) = (1-d\ t)^{-1}-(1+i)^t$ the spread between the factors (where by definition $\xi(t')=0$), with the given rates we obtain: $\xi(1) = 0.000699$. In addition, $\xi(0.5) = -0.000137$. Starting with the *dichotomic method* between t=1 and t=0.50, we obtain: $\zeta(0.750) = 0.000150$; $\zeta(0.625) = -0.000025$; .....

To speed up the convergence, we proceed with the *secant method*, using the upper bound t = 0.750 fixed and the increasing lower bound from t = 0.625.

1st step: linear interpolation between t = 0.625 and t = 0.750:

$$\frac{t-0.625}{0.750-0.625}=\frac{0-\xi(0.625)}{\xi(0.750)-\xi(0.625)}=\frac{25}{150+25}=0.142857$$

from which $t = 0.625 + 0.125 \cdot 0.142857 = 0.642857$; $\xi(t) = -0.000004$.

2nd step: linear interpolation between t = 0.642857 and t = 0.750:

$$\frac{t-0.642857}{0.750-0.642857}=\frac{0-\xi(0.642857)}{\xi(0.750)-\xi(0.642857)}=\frac{4}{150+4}=0{,}025974$$

from which: $t = 0.642857 + 0.107143 \cdot 0.025974 = 0.645640$; $\xi(t) = -0.000001$.

We stop here: time 0.645640 = (7m+22d) is a good estimation (approximated by default) of the indifference length t', because the spread ξ is close to zero.

With reference to *discount laws*, for the problem of

B1) *comparison between RD and CCD laws*;

B2) *comparison between RD and SD laws*;

B3) *comparison between SD and CCD laws*.

we obtain the same indifference length t' valid for the interest conjugate laws, as it is simple to prove by observing that the equations giving the solutions concern the reciprocals of the terms which appear in equations (3.59), (3.60), (3.61) and then coincide with the aforementioned relations.

Furthermore, for length $t \neq t'$, going from interest laws to their conjugated discount laws, the preference relations are inverted[20].

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20 In fact the discount laws give rise to factors reciprocal to those of the interest laws conjugated with the previous discount laws. Therefore, the inequalities and the sign of concavities of the corresponding graphs are inverted. In addition, considering discount, in the right derivatives in t=0 only the sign changes, i.e. there are $-i_0$, -d, $-d$. This is in agreement with the generally valid property, that the differentiable functions $f(x)$ and their reciprocal function have in the intersection points opposite derivatives. Indeed, if $f(x_0) = 1/f(x_0)$, we obtain: $[f(x_0)]^2 = 1$ and then $1/f(x)_{x=x_0} = -f\,'(x_0)$.